GENERALIZED REPEATED INTERACTION MODEL AND TRANSFER FUNCTIONS

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Abstract

Using a scheme involving a lifting of a row contraction we introduce a toy model of repeated interactions between quantum systems. In this model there is an outgoing Cuntz scattering system involving two wandering subspaces. We associate to this model an input/output linear system which leads to a transfer function. This transfer function is a multi-analytic operator, and we show that it is inner if we assume that the system is observable. Finally it is established that transfer functions coincide with characteristic functions of associated liftings.

Key words: repeated interaction, quantum system, multivariate operator theory, row contraction, contractive lifting, outgoing Cuntz scattering system, transfer function, multi-analytic operator, input-output formalism, linear system, observability, scattering theory, characteristic function.
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1 Introduction

In the article [Go10] the author has commented that an integration of schemes, by replacing the assumption of a one dimensional common eigenspace (corresponding to the vacuum vector) of the associated operator tuple in the toy model of [Go10] with a canonical construction involving a lifting of a row contraction in the setting of [DG11], in future may help to remove unnecessarily restrictive assumptions of their model. This paper attempts to achieve some of these objectives. In the model of repeated interactions between quantum systems, also called a noncommutative Markov chain, studied in [Go10] (cf. [Go04]) for given three Hilbert spaces $\mathcal{H}, \mathcal{K}$ and $\mathcal{P}$ with unit vectors $\Omega^{\mathcal{H}}, \Omega^{\mathcal{K}}$ and $\Omega^{\mathcal{P}}$ an *interaction* is defined to be a unitary operator $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{P}$ such that

$$U(\Omega^{\mathcal{H}} \otimes \Omega^{\mathcal{K}}) = \Omega^{\mathcal{H}} \otimes \Omega^{\mathcal{P}}. \tag{1.1}$$

Define $\mathcal{K}_\infty := \bigotimes_{i=1}^{\infty} \mathcal{K}$ and $\mathcal{P}_\infty := \bigotimes_{i=1}^{\infty} \mathcal{P}$ as infinite tensor products of Hilbert spaces with distinguished unit vectors. We denote m-th copies of $\mathcal{K}_\infty$ by $\mathcal{K}_m$ and set $\mathcal{K}_{[m,n]} := \mathcal{K}_m \otimes \cdots \otimes \mathcal{K}_n$. Similar notations are also used with respect to $\mathcal{P}$. The repeated interaction is defined as

$$U(n) := U_n \dots U_1 : \mathcal{H} \otimes \mathcal{K}_\infty \to \mathcal{H} \otimes \mathcal{P}_{[1,n]} \otimes \mathcal{K}_{[n+1,\infty)}$$

where U_i's are copies of U on the factors $\mathcal{H} \otimes \mathcal{K}_i$ of the infinite tensor products and U_i's leaves other factors fixed. Equation (1.1) tells us that the tensor product of the vacuum vectors $\Omega^{\mathcal{H}}, \Omega^{\mathcal{K}}$ (along with $\Omega^{\mathcal{P}}$) represents a state of the coupled system which is not affected by the interaction U. This entire setting represents interactions of an atom with light beams or fields. In particular $\Omega^{\mathcal{H}}$ in [Go10] is thought of as the vacuum state of an atom, and $\Omega^{\mathcal{K}}$ and $\Omega^{\mathcal{P}}$ as a state indicating the absence of photons.

In the generalized repeated interaction model that we introduce in this article we use a pair of unitaries to encode the interactions instead of one unitary as follows:
Let $\tilde{\mathcal{H}}$ be a closed subspace of $\mathcal{H}$, and $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{P}$ and $\tilde{U} : \tilde{\mathcal{H}} \otimes \mathcal{K} \to \tilde{\mathcal{H}} \otimes \mathcal{P}$ be two unitaries such that

$$U(\tilde{h} \otimes \Omega^{\mathcal{K}}) = \tilde{U}(\tilde{h} \otimes \Omega^{\mathcal{K}}) \text{ for all } \tilde{h} \in \tilde{\mathcal{H}}. \tag{1.2}$$

We fix $\{\epsilon_1, \dots, \epsilon_d\}$ to be an orthonormal basis of $\mathcal{P}$. The equation (1.2) is the analog of the equation (1.1) for our model and thus our model can be used to describe some model for quantum system involving a stream of atoms interacting with a light beam and a fixed generalized state, i.e., a completely positive map. In this article we are interested in exploiting the rich mathematical structure of this model.

The focus of the study done here, as also in [Go10], is to bring out that certain multi-analytic operators of the multivariate operator theory are associated to noncommutative Markov chains and related models, and these operators can be exploited as powerful tools. These operators occur as central objects in various context such as in the systems theory related works (cf. [BV05]) and noncommutative multivariable operator theory related works (cf. [Po89b], [Po95]).

A tuple $\underline{T} = (T_1, \dots, T_d)$ of operators T_i's on a common Hilbert space $\mathcal{L}$ is called a *row contraction* if $\sum_{i=1}^d T_i T_i^* \leq I$. In particular if $\sum_{i=1}^d T_i T_i^* = I$, then the tuple $\underline{T} = (T_1, \dots, T_d)$ is called *coisometric*. We introduce the notation $\tilde{\Lambda}$ for the free semigroup with generators $1, \dots, d$. Suppose $T_1, \dots, T_d \in \mathcal{B}(\mathcal{L})$ for a Hilbert space $\mathcal{L}$. If $\alpha \in \tilde{\Lambda}$ is the word $\alpha_1 \dots \alpha_n$ with length $|\alpha| = n$, where each $\alpha_j \in \{1, \dots, d\}$, then T_α denote $T_{\alpha_1} \dots T_{\alpha_n}$. For the empty word $\emptyset$ we define $|\emptyset| = 0$ and $T_\emptyset = I$.

The unitary $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{P}$ from our model can be decomposed as

$$U(h \otimes \Omega^{\mathcal{K}}) = \sum_{j=1}^{d} E_j^* h \otimes \epsilon_j \quad \text{for } h \in \mathcal{H}, \tag{1.3}$$

where E_j's are some operators in $\mathcal{B}(\mathcal{H})$, for $j = 1, \dots, d$. Likewise there exist some operators C_j's in $\mathcal{B}(\tilde{\mathcal{H}})$ such that

$$\tilde{U}(\tilde{h} \otimes \Omega^{\mathcal{K}}) = \sum_{j=1}^{d} C_j^* \tilde{h} \otimes \epsilon_j \quad \text{for } \tilde{h} \in \tilde{\mathcal{H}}. \tag{1.4}$$

Observe that $\sum_{j=1}^d E_j E_j^* = I$ and $\sum_{j=1}^d C_j C_j^* = I$, i.e., $\underline{E}$ and $\underline{C}$ are coisometric tuples. By equation (1.2)

$$E_j^* \tilde{h} = C_j^* \tilde{h} \quad \text{for all } \tilde{h} \in \tilde{\mathcal{H}}.$$

We recall from [DG11] that such tuple $\underline{E} = (E_1, \dots, E_d)$ is called a *lifting* of $\underline{C} = (C_1, \dots, C_d)$. Our model replaces the vector state $X \to \langle \Omega^{\mathcal{H}}, X \Omega^{\mathcal{H}} \rangle$ on $\mathcal{B}(\mathcal{H})$ of [Go10] corresponding to the vacuum vector $\Omega^{\mathcal{H}}$ by the unital completely positive map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\tilde{\mathcal{H}})$ defined by

$$\phi(X) := P_{\tilde{\mathcal{H}}} X|_{\tilde{\mathcal{H}}} \quad \text{for all } X \in \mathcal{B}(\mathcal{H}).$$

Note that

$$\phi(E_\alpha (E_\beta)^*) = C_\alpha (C_\beta)^* \quad \text{for all } \alpha, \beta \in \tilde{\Lambda}.$$

In section 2 we develop our generalized repeated interaction model and obtain a coisometric operator which intertwines between the minimal isometric dilations of $\underline{E}$ and $\underline{C}$, and which will be a crucial tool for the further investigation in this article. Using this an outgoing Cuntz scattering system in the sense of [BV05] is constructed for our model in section 3. Popescu introduced the minimal isometric dilation in [Po89a] and the characteristic function in [Po89b] of a row contraction, and systematically developed an extensive theory of row contractions (cf. [Po99], [Po06]). We use some of the concepts from Popescu's theory in this work.

For the outgoing Cuntz scattering system in section 4 we give a $\tilde{\Lambda}$-linear system with an input-output formalism. A multi-analytic operator appear here as the transfer function and we are benefited in our effort to understand these transfer functions on employing the language of power series for them. In [YK03] and [GGY08] there are other approaches to transfer functions. Several works on transfer functions and on quantum systems using linear system theory can be found in recent theoretical physics and control theory surveys. In section 5 we investigate in regard to our model what the notion of observability implies for the scattering theory and the theory of liftings. Characteristic functions for liftings, introduced in [DG11], are multi-analytic operators which classify certain class of liftings. Our model is a vast generalization of the setting of [Go10], and a comparison is done in section 6 between the transfer function of our model and the characteristic function for the associated lifting using the series expansion of the transfer function obtained in section 4.

2 A Generalised repeated Interaction Model

We begin with three Hilbert spaces $\mathcal{H}, \mathcal{K}$ and $\mathcal{P}$ with unit vectors $\Omega^{\mathcal{K}} \in \mathcal{K}$ and $\Omega^{\mathcal{P}} \in \mathcal{P}$, and unitaries U and $\tilde{U}$ as in equation (1.2). In $\mathcal{K}_\infty = \bigotimes_{i=1}^\infty \mathcal{K}$ and $\mathcal{P}_\infty = \bigotimes_{i=1}^\infty \mathcal{P}$ define $\Omega^{\mathcal{K}}_\infty := \bigotimes_{i=1}^\infty \Omega^{\mathcal{K}}$ and $\Omega^{\mathcal{P}}_\infty := \bigotimes_{i=1}^\infty \Omega^{\mathcal{P}}$ respectively. We denote m-th copies of $\Omega^{\mathcal{K}}_\infty$ by $\Omega^{\mathcal{K}}_m$ and in terms of this we introduce the notation $\Omega^{\mathcal{K}}_{[m,n]} := \Omega^{\mathcal{K}}_m \otimes \cdots \otimes \Omega^{\mathcal{K}}_n$. Identify $\mathcal{K}_{[m,n]}$ with $\Omega^{\mathcal{K}}_{[1,m-1]} \otimes \mathcal{K}_{[m,n]} \otimes \Omega^{\mathcal{K}}_{[n+1,\infty)}$, $\mathcal{H}$ with $\mathcal{H} \otimes \Omega^{\mathcal{K}}_\infty$ as a subspace of $\mathcal{H} \otimes \mathcal{K}_\infty$ and $\tilde{\mathcal{H}}$ with $\tilde{\mathcal{H}} \otimes \Omega^{\mathcal{K}}_\infty$ as a subspace of $\tilde{\mathcal{H}} \otimes \mathcal{K}_\infty$. Similar notations with respect to $\mathcal{P}$ are also used. For simplicity we assume that d is finite but all the results here can be derived also for $d = \infty$.

Define isometries

$$\widehat{V}^E_j(h \otimes \eta) := U^*(h \otimes \epsilon_j) \otimes \eta \text{ for } j = 1, \dots, d,$$

on the elementary tensors $h \otimes \eta \in \mathcal{H} \otimes \mathcal{K}_\infty$ and extend it linearly to obtain $\widehat{V}^E_j \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}_\infty)$ for $j = 1, \dots, d$. We recall that a lifting $\underline{T} = (T_1, \dots, T_d)$ of any row contraction $\underline{S} = (S_1, \dots, S_d)$ is called its isometric dilation if T_i's are isometries. It can be easily verified that $\underline{\widehat{V}}^E = (\widehat{V}^E_1, \dots, \widehat{V}^E_d)$ on the space $\mathcal{H} \otimes \mathcal{K}_\infty$ is an isometric dilation of $\underline{E} = (E_1, \dots, E_d)$ (cf. section 1 of [DG07]). If $h \in \mathcal{H}$ and $k_1 \in \mathcal{K}$, then there exist $h_i \in \mathcal{H}$ for $i = 1, \dots, d$ such that $U^*(\sum_{i=1}^d h_i \otimes \epsilon_i) = h \otimes k_1$ because U is a unitary. This implies

$$\sum_{i=1}^d \widehat{V}^E_i (h_i \otimes \Omega^{\mathcal{K}}_\infty) = h \otimes k_1 \otimes \Omega^{\mathcal{K}}_{[2,\infty)}.$$

In addition if $k_2 \in \mathcal{K}$, then

$$\sum_{i=1}^d \widehat{V}^E_i (h_i \otimes k_2 \otimes \Omega^{\mathcal{K}}_{[2,\infty)}) = U^*(\sum_{i=1}^d h_i \otimes \epsilon_i) \otimes k_2 \otimes \Omega^{\mathcal{K}}_{[3,\infty)} = h \otimes k_1 \otimes k_2 \otimes \Omega^{\mathcal{K}}_{[3,\infty)}.$$

By induction we conclude that

$$\mathcal{H}\otimes\mathcal{K}_\infty=\overline{span}\{\widehat{V}^E_\alpha(h\otimes\Omega^{\mathcal{K}}_\infty):h\in\mathcal{H},\alpha\in\tilde{\Lambda}\},$$

i.e., $\underline{\widehat{V}}^E$ is the minimal isometric dilation of $\underline{E}$. Note that the minimal isometric dilation is unique up to unitary equivalence (cf. [Po89a]).

Similarly, define isometries

$$\widehat{V}^C_j(\tilde{h}\otimes\eta):=\tilde{U}^*(\tilde{h}\otimes\epsilon_j)\otimes\eta \text{ for } j=1,\dots,d$$

on the elementary tensors $\tilde{h}\otimes\eta\in\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty$ and extend it linearly to obtain $\widehat{V}^C_j\in\mathcal{B}(\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty)$ for $j=1,\dots,d$. The tuple $\underline{\widehat{V}}^C=(\widehat{V}^C_1,\dots,\widehat{V}^C_d)$ on the space $\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty$ is the minimal isometric dilation of $\underline{C}=(C_1,\dots,C_d)$. Recall that

$$U_m:\mathcal{H}\otimes\mathcal{K}_\infty\to\mathcal{H}\otimes\mathcal{K}_{[1,m-1]}\otimes\mathcal{P}_m\otimes\mathcal{K}_{[m+1,\infty)}$$

is nothing but the operator which acts as U on $\mathcal{H}\otimes\mathcal{K}_m$ and fixes other factors of the infinite tensor products. Similarly, we define $\tilde{U}_m$ using $\tilde{U}$.

Proposition 2.1. *Let $P_n:=P_{\tilde{\mathcal{H}}}\otimes I_{\mathcal{P}_{[1,n]}}\otimes I_{\mathcal{K}_{[n+1,\infty)}}\in B(\mathcal{H}\otimes\mathcal{P}_{[1,n]}\otimes\mathcal{K}_{[n+1,\infty)})$ for $n\in\mathbb{N}$. Then*

$$sot-\lim_{n\to\infty}\tilde{U}_1^*\dots\tilde{U}_n^*P_nU_n\dots U_1$$

exists and this limit defines a coisometry $\widehat{W}:\mathcal{H}\otimes\mathcal{K}_\infty\to\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty$. Its adjoint $\widehat{W}^*:\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty\to\mathcal{H}\otimes\mathcal{K}_\infty$ is given by

$$\widehat{W}^*=sot-\lim_{n\to\infty}U_1^*\dots U_n^*\tilde{U}_n\dots\tilde{U}_1.$$

Here sot stands for the strong operator topology.

Proof. At first we construct the adjoint $\widehat{W}^*$. For that consider a dense subspace $\bigcup_{m\geq 1}\tilde{\mathcal{H}}\otimes\mathcal{K}_{[1,m]}$ of $\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty$ and let an arbitrary simple tensor element of this dense subspace be $\tilde{h}\otimes k_1\otimes\dots\otimes k_\ell\otimes\Omega^{\mathcal{K}}_{[\ell+1,\infty)}$ for some $\ell\in\mathbb{N},\tilde{h}\in\tilde{\mathcal{H}}$ and $k_i\in\mathcal{K}_i$. Set $a_p=U_1^*\dots U_p^*\tilde{U}_p\dots\tilde{U}_1(\tilde{h}\otimes k_1\otimes\dots\otimes k_\ell\otimes\Omega^{\mathcal{K}}_{[\ell+1,\infty)})$ for $p\in\mathbb{N}$. Since $U(\tilde{h}\otimes\Omega^{\mathcal{K}})=\tilde{U}(\tilde{h}\otimes\Omega^{\mathcal{K}})$ for all $\tilde{h}\in\tilde{\mathcal{H}}$, we have $a_\ell=a_{\ell+n}$ for all $n\in\mathbb{N}$. Therefore we deduce that

$$\lim_{n\to\infty}U_1^*\dots U_n^*\tilde{U}_n\dots\tilde{U}_1(\tilde{h}\otimes k_1\otimes\dots\otimes k_\ell\otimes\Omega^{\mathcal{K}}_{[\ell+1,\infty)})$$

exists. Because U and $\tilde{U}$ are unitaries, we obtain an isometric extension $\widehat{W}^*$ to the whole of $\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty$. Thus its adjoint is a coisometry $\widehat{W}:\mathcal{H}\otimes\mathcal{K}_\infty\to\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty$.

Now we will derive the limit form for $\widehat{W}$ as claimed in the statement of the proposition. If $h\otimes\eta\in\mathcal{H}\otimes\mathcal{K}_{[1,k]},\tilde{h}\otimes\tilde{\eta}\in\tilde{\mathcal{H}}\otimes\mathcal{K}_{[1,n]}$ and $k\leq n$, then

$$\begin{aligned}\langle\widehat{W}(h\otimes\eta),\tilde{h}\otimes\tilde{\eta}\rangle &= \langle h\otimes\tilde{\eta},\widehat{W}^*(\tilde{h}\otimes\tilde{\eta})\rangle\\ &= \langle h\otimes\eta,U_1^*\dots U_n^*\tilde{U}_n\dots\tilde{U}_1(\tilde{h}\otimes\tilde{\eta})\rangle\\ &= \langle\tilde{U}_1^*\dots\tilde{U}_n^*P_nU_n\dots U_1(h\otimes\eta),\tilde{h}\otimes\tilde{\eta}\rangle.\end{aligned}$$

Consequently $\widehat{W}=sot-\lim_{n\to\infty}\tilde{U}_1^*\dots\tilde{U}_n^*P_nU_n\dots U_1$ on a dense subspace and therefore it can be extended to the whole of $\mathcal{H}\otimes\mathcal{K}_\infty$. □

Observe that

$$\widehat{W}^*(\tilde{h} \otimes \Omega_\infty^{\mathcal{K}}) = \tilde{h} \otimes \Omega_\infty^{\mathcal{K}} \text{ for all } \tilde{h} \in \tilde{\mathcal{H}}. \tag{2.1}$$

Next we show that this coisometry $\widehat{W}$ intertwines between $\widehat{V}_j^E$ and $\widehat{V}_j^C$ for all $j = 1, \ldots, d$. For $j = 1, \ldots, d$, define

$$\begin{aligned} S_j : \mathcal{H} \otimes \mathcal{K}_\infty &\to \mathcal{H} \otimes \mathcal{P}_1 \otimes \mathcal{K}_{[2,\infty)}, \\ h \otimes \eta &\mapsto h \otimes \epsilon_j \otimes \eta. \end{aligned}$$

The following are immediate:

(1) $S_j^*(h \otimes p_1 \otimes \eta) = \langle \epsilon_j, p_1 \rangle (h \otimes \eta)$ for $(h \otimes p_1 \otimes \eta) \in \mathcal{H} \otimes \mathcal{P}_1 \otimes \mathcal{K}_{[2,\infty)}$.

(2) $\widehat{V}_j^E(h \otimes \eta) = U_1^* S_j(h \otimes \eta)$ for $h \otimes \eta \in \mathcal{H} \otimes \mathcal{K}_\infty$.

(3) $\widehat{V}_j^C(\tilde{h} \otimes \eta) = \tilde{U}_1^* S_j(\tilde{h} \otimes \eta)$ for $\tilde{h} \otimes \eta \in \tilde{\mathcal{H}} \otimes \mathcal{K}_\infty$.

Proposition 2.2. *If $\widehat{W}$ is as in Proposition 2.1, then*

$$\widehat{W}\widehat{V}_j^E = \widehat{V}_j^C \widehat{W}, \quad \widehat{V}_j^E \widehat{W}^* = \widehat{W}^* \widehat{V}_j^C \text{ for all } j = 1, \ldots, d.$$

Proof. If $h \in \mathcal{H}, \eta \in \mathcal{K}_\infty, \tilde{h} \in \tilde{\mathcal{H}}$ and $k_i \in \mathcal{K}_i$, then by the three observations that were noted preceding this proposition

$$\begin{aligned} & \langle\, \widehat{W}\widehat{V}_j^E(h \otimes \eta),\ \tilde{h} \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}} \,\rangle \\ = & \langle\, U^*(h \otimes \epsilon_j) \otimes \eta,\ U_1^* \ldots U_\ell^* \tilde{U}_\ell \ldots \tilde{U}_1(\tilde{h} \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}}) \,\rangle. \end{aligned}$$

Substituting $\tilde{U}(\tilde{h} \otimes k_1) = \sum_i \tilde{h}^{(i)} \otimes k_1^{(i)}$ where $\tilde{h}^{(i)} \in \tilde{\mathcal{H}}$ and $k_1^{(i)} \in \mathcal{K}$ we obtain

$$\begin{aligned} & \langle\, \widehat{W}\widehat{V}_j^E(h \otimes \eta),\ \tilde{h} \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}} \,\rangle \\ = & \langle\, h \otimes \epsilon_j \otimes \eta,\ U_2^* \ldots U_\ell^* \tilde{U}_\ell \ldots \tilde{U}_2(\sum_i (\tilde{h}^{(i)} \otimes k_1^{(i)}) \otimes k_2 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}}) \,\rangle \\ = & \sum_i \langle \epsilon_j,\ k_1^{(i)} \rangle \,\langle\, h \otimes \eta,\ \widehat{W}^*(\tilde{h}^{(i)} \otimes k_2 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}}) \,\rangle \\ = & \langle\, \widehat{W}(h \otimes \eta),\ S_j^* \tilde{U}_1(\tilde{h} \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}}) \,\rangle \\ = & \langle\, \tilde{U}_1^* S_j \widehat{W}((h \otimes \eta),\ \tilde{h} \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}} \,\rangle \\ = & \langle\, \widehat{V}_j^C \widehat{W}(h \otimes \eta),\ \tilde{h} \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}} \,\rangle. \end{aligned}$$

Hence $\widehat{W}\widehat{V}_j^E = \widehat{V}_j^C \widehat{W}$ for all $j = 1, \ldots, d$. To obtain the other equation of the proposition we again use the last two of the three observations as follows:

$$\begin{aligned} & \widehat{W}^* \widehat{V}_j^C(\tilde{h} \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}}) \\ = & \widehat{W}^* \tilde{U}_1^*(\tilde{h} \otimes \epsilon_j \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+2,\infty)}^{\mathcal{K}}) \\ = & U_1^* U_2^* \ldots U_{\ell+1}^* \tilde{U}_{\ell+1} \ldots \tilde{U}_2 \tilde{U}_1 \tilde{U}_\ell^*(\tilde{h} \otimes \epsilon_j \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+2,\infty)}^{\mathcal{K}}) \\ = & U_1^* U_2^* \ldots U_{\ell+1}^* \tilde{U}_{\ell+1} \ldots \tilde{U}_2 S_j(\tilde{h} \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}}) \\ = & U_1^* S_j U_1^* \ldots U_\ell^* \tilde{U}_\ell \ldots \tilde{U}_1(\tilde{h} \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}}) \\ = & \widehat{V}_j^E \widehat{W}^*(\tilde{h} \otimes k_1 \otimes \ldots \otimes k_\ell \otimes \Omega_{[\ell+1,\infty)}^{\mathcal{K}}) \end{aligned}$$

□

Further define

$$(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ := (\mathcal{H}\otimes\mathcal{K}_\infty)\ominus(\tilde{\mathcal{H}}\otimes\Omega_\infty^{\mathcal{K}}),$$
$$(\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty)^\circ := (\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty)\ominus(\tilde{\mathcal{H}}\otimes\Omega_\infty^{\mathcal{K}}) \text{ and } \mathcal{H}^\circ := \mathcal{H}\ominus\tilde{\mathcal{H}}.$$

Let $\sum_{i=1}^k \xi_i\otimes\eta_i \in (\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$ and $\tilde{h}\in\tilde{\mathcal{H}}$. Then for $j=1,\dots,d$

$$\begin{aligned}\langle \widehat{V}_j^E(\sum_i \xi_i\otimes\eta_i), \tilde{h}\otimes\Omega_\infty^{\mathcal{K}}\rangle &= \langle \sum_i U^*(\xi_i\otimes\epsilon_j)\otimes\eta_i, \tilde{h}\otimes\Omega_\infty^{\mathcal{K}}\rangle \\ &= \langle \sum_i \xi_i\otimes\epsilon_j\otimes\eta_i, \tilde{U}(\tilde{h}\otimes\Omega_1^{\mathcal{K}})\otimes\Omega_{[2,\infty)}^{\mathcal{K}}\rangle = 0\end{aligned}$$

because $\tilde{U}$ maps into $\tilde{\mathcal{H}}\otimes\mathcal{P}$ and $\sum_{i=1}^k \xi_i\otimes\eta_i \perp \tilde{\mathcal{H}}\otimes\Omega^{\mathcal{K}}$. Therefore $\widehat{V}_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ \subset (\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$ for $j=1,\dots,d$. Similarly $\widehat{V}_j^C(\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty)^\circ \subset (\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty)^\circ$ for $j=1,\dots,d$. Set $V_j^E := \widehat{V}_j^E|_{(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ}$ and $V_j^C := \widehat{V}_j^C|_{(\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty)^\circ}$ for $j=1,\dots,d$. If we define

$$W^* := \widehat{W}^*|_{(\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty)^\circ},$$

then by equation (2.1) it follows that $W^*\in\mathcal{B}((\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty)^\circ, (\mathcal{H}\otimes\mathcal{K}_\infty)^\circ)$. The operator W^* is an isometry because it is a restriction of an isometry and W, the adjoint of W^*, is the restriction of $\widehat{W}$ to $(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$, i.e., $W=\widehat{W}|_{(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ}$.

Remark 2.3. *It follows that*

$$WV_j^E = V_j^C W$$

for $j=1,\dots,d$.

3 Outgoing Cuntz Scattering Systems

In this section we aim to construct an outgoing Cuntz scattering system (cf. [BV05]) for our model. This will assist us in the next section to work with an input-output formalism and to associate a transfer function to the model.

Following are some notions from the multivariable operator theory:

Definition 3.1. *Suppose $\underline{T}=(T_1,\dots,T_d)$ is a row contraction where $T_i\in\mathcal{B}(\mathcal{L})$.*

(1) If T_i's are isometries with orthogonal ranges, then the tuple $\underline{T}=(T_1,\dots,T_d)$ is called a row isometry.

(2) If $\overline{span}_{j=1,\dots,d}T_j\mathcal{L}=\mathcal{L}$ and $\underline{T}=(T_1,\dots,T_d)$ is a row isometry, then $\underline{T}$ is called a row unitary.

(3) If there exist a subspace $\mathcal{E}$ of $\mathcal{L}$ such that $\mathcal{L}=\oplus_{\alpha\in\tilde{\Lambda}}T_\alpha\mathcal{E}$ and $\underline{T}=(T_1,\dots,T_d)$ is a row isometry, then $\underline{T}$ is called a row shift and $\mathcal{E}$ is called a wandering subspace of $\mathcal{L}$ w.r.t. $\underline{T}$.

Definition 3.2. *A collection $(\mathcal{L},\underline{V}=(V_1,\dots,V_d),\mathcal{G}_*^+,\mathcal{G})$ is called an outgoing Cuntz scattering system (cf. [BV05]), if $\underline{V}$ is a row isometry on the Hilbert space $\mathcal{L}$, and $\mathcal{G}_*^+$ and $\mathcal{G}$ are subspaces of $\mathcal{L}$ such that*

(1) for $\mathcal{E}_* := \mathcal{L}\ominus\overline{span}_{j=1,\dots,d}V_j\mathcal{L}$, the tuple $\underline{V}|\mathcal{G}_*^+$ is a row shift where $\mathcal{G}_*^+=\bigoplus_{\alpha\in\tilde{\Lambda}}V_\alpha\mathcal{E}_*$.

(2) there exist $\mathcal{E} := \mathcal{G}\ominus\overline{span}_{j=1,\dots,d}V_j\mathcal{G}$ with $\mathcal{G}=\bigoplus_{\alpha\in\tilde{\Lambda}}V_\alpha\mathcal{E}$, i.e., $\underline{V}|_{\mathcal{G}}$ is a row shift.

We continue using the notations from the previous section. $\widehat{V}_j^E$'s are isometries with orthogonal ranges and because $(\epsilon_j)_{j=1}^d$ is an orthonormal basis of $\mathcal{P}$, we have

$$\overline{span}_{j=1,\ldots,d}\widehat{V}_j^E(\mathcal{H}\otimes\mathcal{K}_\infty) = \mathcal{H}\otimes\mathcal{K}_\infty.$$

Thus $\underline{\widehat{V}}^E$ is a row unitary on $\mathcal{H}\otimes\mathcal{K}_\infty$. Now using the fact that $V_j^E = \widehat{V}_j^E|_{(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ}$ we infer that V_j^E's are isometries with orthogonal ranges. Therefore $\underline{V}^E$ is a row isometry on $(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$.

Proposition 3.3. *If* $\mathcal{Y} := \tilde{\mathcal{H}}\otimes(\Omega_1^{\mathcal{K}})^\perp\otimes\Omega_{[2,\infty)}^{\mathcal{K}} \subset \tilde{\mathcal{H}}\otimes\mathcal{K}_\infty$*, then*

$$W^*\mathcal{Y} \perp \overline{span}_{j=1,\ldots,d}\ V_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ.$$

Proof. By Proposition 2.1 it is easy to see that

$$W^*\mathcal{Y} = U_1^*\tilde{U}_1\mathcal{Y} \subset \mathcal{H}\otimes\mathcal{K}_1\otimes\Omega_{[2,\infty)}^{\mathcal{K}}. \tag{3.1}$$

Let $\tilde{h}_i \in \tilde{\mathcal{H}}$ and $k_i \perp \Omega_1^{\mathcal{K}}$ for $i = 1,\ldots,n$, i.e., $\sum_i \tilde{h}_i\otimes k_i\otimes\Omega_{[2,\infty)}^{\mathcal{K}} \in \mathcal{Y}$. For $\sum_k h_k\otimes\eta_k \in (\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$ with $h_k\in\mathcal{H}$ and $\eta_k\in\mathcal{K}_\infty$

$$\begin{aligned}
&\langle\ W^*(\textstyle\sum_i \tilde{h}_i\otimes k_i\otimes\Omega_{[2,\infty)}^{\mathcal{K}}), V_j^E(\sum_k h_k\otimes\eta_k)\ \rangle \\
=\ &\langle\ U^*\tilde{U}(\textstyle\sum_i \tilde{h}_i\otimes k_i)\otimes\Omega_{[2,\infty)}^{\mathcal{K}}, \sum_k U^*(h_k\otimes\epsilon_j)\otimes\eta_k\ \rangle \\
=\ &\langle\ \tilde{U}(\textstyle\sum_i \tilde{h}_i\otimes k_i)\otimes\Omega_{[2,\infty)}^{\mathcal{K}}, \sum_k h_k\otimes\epsilon_j\otimes\eta_k\ \rangle = 0.
\end{aligned}$$

The last equality holds because $\sum_k h_k\otimes\eta_k \perp \tilde{\mathcal{H}}\otimes\Omega_\infty^{\mathcal{K}}$. Thus $W^*\mathcal{Y} \perp \overline{span}_{j=1,\ldots,d}V_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$. ☐

Proposition 3.4. *If $\mathcal{Y}$ is defined as in the previous proposition, then $W^*\mathcal{Y}$ is a wandering subspace of $\underline{V}^E$, i.e., $V_\alpha^E(W^*\mathcal{Y}) \perp V_\beta^E(W^*\mathcal{Y})$ whenever $\alpha,\beta\in\tilde{\Lambda}$, $\alpha\neq\beta$, and*

$$W^*\mathcal{Y} = (\mathcal{H}\otimes\mathcal{K}_\infty)^\circ \ominus \overline{span}_{j=1,\ldots,d}V_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ.$$

Proof. By Proposition 3.3 it is immediate that $V_\alpha^E(W^*\mathcal{Y}) \perp V_\beta^E(W^*\mathcal{Y})$ whenever $\alpha,\beta\in\tilde{\Lambda}$, $\alpha\neq\beta$ and $W^*\mathcal{Y} \subset (\mathcal{H}\otimes\mathcal{K}_\infty)^\circ \ominus \overline{span}_{j=1,\ldots,d}V_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$. The only thing that remains to be shown is that

$$(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ \ominus \overline{span}_{j=1,\ldots,d}V_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ \subset W^*\mathcal{Y}.$$

Let $x \in (\mathcal{H}\otimes\mathcal{K}_\infty)^\circ\ominus\overline{span}_{j=1,\ldots,d}V_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$. Write down the decomposition of x as $x_1\oplus x_2$ w.r.t. $W^*\mathcal{Y}\oplus(W^*\mathcal{Y})^\perp$. So $x - x_1 = x_2$ is orthogonal to both $\overline{span}_{j=1,\ldots,d}V_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$ and $W^*\mathcal{Y}$. Now we show that if any element in $(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$ is orthogonal to $\overline{span}_{j=1,\ldots,d}V_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$ and $W^*\mathcal{Y}$, then it is the zero vector. Let x_0 be such an element. Because $x_0 \in (\mathcal{H}\otimes\mathcal{K}_\infty)^\circ$ and $x_0 \perp W^*\mathcal{Y}$,

$$x_0 \perp U^*(\tilde{\mathcal{H}}\otimes\epsilon_j)\otimes\Omega_{[2,\infty)}^{\mathcal{K}}$$

for $j = 1,\ldots,d$. This implies $x_0 \perp \overline{span}_{j=1,\ldots,d}\widehat{V}_j^E(\tilde{\mathcal{H}}\otimes\Omega_\infty^{\mathcal{K}})$. We also know that

$$x_0 \perp \overline{span}_{j=1,\ldots,d}V_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ (= \overline{span}_{j=1,\ldots,d}\widehat{V}_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ).$$

Therefore

$$x_0 \perp \overline{span}_{j=1,\ldots,d}\widehat{V}_j^E(\mathcal{H}\otimes\mathcal{K}_\infty).$$

Since $\underline{\widehat{V}}^E$ is a row unitrary, $x_0 \perp \mathcal{H}\otimes\mathcal{K}_\infty$. So $x_0 = 0$ and hence $x = x_1 \in W^*\mathcal{Y}$. We conclude that $(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ \ominus \overline{span}_{j=1,\ldots,d}V_j^E(\mathcal{H}\otimes\mathcal{K}_\infty)^\circ \subset W^*\mathcal{Y}$. ☐

Proposition 3.5. *If* $\mathcal{E} := \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^{\perp} \otimes \Omega_{[2,\infty)}^{\mathcal{K}} \subset (\mathcal{H} \otimes \mathcal{K}_\infty)^\circ$*, then* $V_\alpha^E \mathcal{E} \perp V_\beta^E \mathcal{E}$ *whenever* $\alpha, \beta \in \tilde{\Lambda}, \alpha \neq \beta$ *and* $(\mathcal{H} \otimes \mathcal{K}_\infty)^\circ = \mathcal{H}^\circ \oplus \bigoplus_{\alpha \in \tilde{\Lambda}} V_\alpha^E \mathcal{E}$*.*

Proof. If $|\alpha| = |\beta|$ and $\alpha \neq \beta$, then it is easy to see that $V_\alpha^E \mathcal{E} \perp V_\beta^E \mathcal{E}$ because ranges of V_j^E's are mutually orthogonal. If $|\alpha| \neq |\beta|$ (without loss of generality we can assume that $|\alpha| > |\beta|$), then by taking the inner product at the tensor factor $\mathcal{K}_{|\alpha|+1}$ we obtain $V_\alpha^E \mathcal{E} \perp V_\beta^E \mathcal{E}$.

To prove the second part of the proposition, observe that for $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{H} \otimes \mathcal{K}_{[1,n]} \otimes \Omega_{[n+1,\infty]}^{\mathcal{K}} &= (\mathcal{H} \otimes \Omega_\infty^{\mathcal{K}}) \oplus (\mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^{\perp} \otimes \Omega_{[2,\infty)}^{\mathcal{K}}) \oplus (\mathcal{H} \otimes \mathcal{K}_1 \otimes (\Omega_2^{\mathcal{K}})^{\perp} \\
&\quad \otimes \Omega_{[3,\infty)}^{\mathcal{K}}) \oplus \cdots \oplus \ (\mathcal{H} \otimes \mathcal{K}_{[1,n-1]} \otimes (\Omega_n^{\mathcal{K}})^{\perp} \otimes \Omega_{[n+1,\infty)}^{\mathcal{K}}) \\
&= (\tilde{\mathcal{H}} \otimes \Omega_\infty^{\mathcal{K}}) \oplus (\mathcal{H}^\circ \otimes \Omega_\infty^{\mathcal{K}}) \oplus \mathcal{E} \oplus \bigoplus_{j=1}^{d} V_j^E \mathcal{E} \oplus \cdots \oplus \bigoplus_{|\alpha|=n-1}^{d} V_\alpha^E \mathcal{E}.
\end{aligned}$$

Taking $n \to \infty$ we have the following:

$$\mathcal{H} \otimes \mathcal{K}_\infty = (\tilde{\mathcal{H}} \otimes \Omega_\infty^{\mathcal{K}}) \oplus (\mathcal{H}^\circ \otimes \Omega_\infty^{\mathcal{K}}) \oplus \bigoplus_{\alpha \in \tilde{\Lambda}} V_\alpha^E \mathcal{E}.$$

Since $(\mathcal{H} \otimes \mathcal{K}_\infty)^\circ = (\mathcal{H} \otimes \mathcal{K}_\infty) \ominus (\tilde{\mathcal{H}} \otimes \Omega_\infty^{\mathcal{K}})$, it follows that

$$(\mathcal{H} \otimes \mathcal{K}_\infty)^\circ = \mathcal{H}^\circ \oplus \bigoplus_{\alpha \in \tilde{\Lambda}} V_\alpha^E \mathcal{E}.$$

□

We sum up Propositions 3.3, 3.4 and 3.5 in the following theorem:

Theorem 3.6. *For a generalized repeated interaction model involving unitaries U and $\tilde{U}$ as before set* $\mathcal{Y} := \tilde{\mathcal{H}} \otimes (\Omega_1^{\mathcal{K}})^{\perp} \otimes \Omega_{[2,\infty)}^{\mathcal{K}}$ *and* $\mathcal{E} := \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^{\perp} \otimes \Omega_{[2,\infty)}^{\mathcal{K}}$*. If* $\mathcal{E}_* := W^* \mathcal{Y}$*,* $\mathcal{G}_*^+ := \bigoplus_{\alpha \in \tilde{\Lambda}} V_\alpha^E \mathcal{E}_*$ *and* $\mathcal{G} := \bigoplus_{\alpha \in \tilde{\Lambda}} V_\alpha^E \mathcal{E}$*, then the collection*

$$((\mathcal{H} \otimes \mathcal{K}_\infty)^\circ, \underline{V}^E = (V_1^E, \ldots, V_d^E), \mathcal{G}_*^+, \mathcal{G})$$

is an outgoing Cuntz scattering system such that $(\mathcal{H} \otimes \mathcal{K}_\infty)^\circ = \mathcal{H}^\circ \oplus \mathcal{G}$*.*

Remark 3.7. *Applying arguments similar to those used for proving the second part of the Proposition 3.5 one can prove the following:*

$$(\tilde{\mathcal{H}} \otimes \mathcal{K}_\infty)^\circ = \bigoplus_{\alpha \in \tilde{\Lambda}} V_\alpha^C \mathcal{Y}.$$

We refer the reader to Proposition 3.1 of [Go11] for a result in a similar direction.

4 $\tilde{\Lambda}$-Linear Systems and Transfer Functions

We would demostrate that the outgoing Cuntz scattering system $((\mathcal{H} \otimes \mathcal{K}_\infty)^\circ, \underline{V}^E = (V_1^E, \ldots, V_d^E), \mathcal{G}_*^+, \mathcal{G})$ from Theorem 3.6 has interesting relations with a generalization of the linear systems theory that is associated to our interaction model. For a given model involving unitaries U and $\tilde{U}$ as before, let us define the input space as

$$\mathcal{U} := \mathcal{E} = \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^{\perp} \otimes \Omega_{[2,\infty)}^{\mathcal{K}} \subset (\mathcal{H} \otimes \mathcal{K}_\infty)^\circ$$

and the output space as

$$\mathcal{Y}=\tilde{\mathcal{H}}\otimes(\Omega_1^{\mathcal{K}})^{\perp}\otimes\Omega^{\mathcal{K}}_{[2,\infty)}\subset(\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty)^\circ.$$

Note that $\mathcal{H}\otimes\mathcal{K}=\mathcal{H}\oplus\mathcal{U}$ and $\tilde{\mathcal{H}}\otimes\mathcal{K}=\tilde{\mathcal{H}}\oplus\mathcal{Y}$. So U maps $\mathcal{H}\oplus\mathcal{U}$ onto $\mathcal{H}\otimes\mathcal{P}$ and $\tilde{U}$ maps $\tilde{\mathcal{H}}\oplus\mathcal{Y}$ onto $\tilde{\mathcal{H}}\otimes\mathcal{P}$. Using unitaries U and $\tilde{U}$ we define $F_j:\mathcal{H}\to\mathcal{U}$ and $D_j:\tilde{\mathcal{H}}\to\mathcal{Y}$ for $j=1,\dots,d$ by

$$\sum_{j=1}^{d}F_j^*\eta\otimes\epsilon_j:=U(0\oplus\eta),\quad \sum_{j=1}^{d}D_j^*y\otimes\epsilon_j:=\tilde{U}(0\oplus y)\ \text{ for } \eta\in\mathcal{U} \text{ and } y\in\mathcal{Y}. \tag{4.1}$$

Combining equation (4.1) with equations (1.3) and (1.4) we have for $h\in\mathcal{H},\ \eta\in\mathcal{U},\ \tilde{h}\in\tilde{\mathcal{H}}$ and $y\in\mathcal{Y}$

$$U(h\oplus\eta)=\sum_{j=1}^{d}(E_j^*h+F_j^*\eta)\otimes\epsilon_j, \tag{4.2}$$

$$\tilde{U}(\tilde{h}\oplus y)=\sum_{j=1}^{d}(C_j^*\tilde{h}+D_j^*y)\otimes\epsilon_j \tag{4.3}$$

respectively. Using equation (4.3) it can be checked that

$$\tilde{U}^*(\tilde{h}\otimes\epsilon_j)=((C_j\tilde{h}\otimes\Omega^{\mathcal{K}})\oplus D_j\tilde{h})\ \text{ for } \tilde{h}\in\tilde{\mathcal{H}}; j=1,\dots,d. \tag{4.4}$$

Let us define

$$\tilde{C}:=\sum_{j=1}^{d}D_jP_{\tilde{\mathcal{H}}}E_j^*:\mathcal{H}\to\mathcal{Y},\quad \tilde{D}:=\sum_{j=1}^{d}D_jP_{\tilde{\mathcal{H}}}F_j^*:\mathcal{U}\to\mathcal{Y}$$

where $P_{\tilde{\mathcal{H}}}$ is the orthogonal projection onto $\tilde{\mathcal{H}}$. It follows that

$$P_{\mathcal{Y}}\tilde{U}^*P_1U(h\oplus\eta)=\tilde{C}h+\tilde{D}\eta \tag{4.5}$$

where $h\in\mathcal{H},\eta\in\mathcal{U},P_1$ is as in Proposition 2.1 and $P_{\mathcal{Y}}$ is the orthogonal projection onto $\mathcal{Y}$. Define a *colligation* of operators (cf. [BV05]) using the operators E_j^*'s, F_j^*'s, $\tilde{C}$ and $\tilde{D}$ by

$$\mathcal{C}_{U,\tilde{U}}:=\begin{pmatrix}E_1^* & F_1^*\\ \vdots & \vdots\\ E_d^* & F_d^*\\ \tilde{C} & \tilde{D}\end{pmatrix}:\mathcal{H}\oplus\mathcal{U}\to\bigoplus_{j=1}^{d}\mathcal{H}\oplus\mathcal{Y}.$$

From the colligation $\mathcal{C}_{U,\tilde{U}}$ we get the following $\tilde{\Lambda}$-*linear system* $\sum_{U,\tilde{U}}$:

$$x(j\alpha)=E_j^*x(\alpha)+F_j^*u(\alpha), \tag{4.6}$$

$$y(\alpha)=\tilde{C}x(\alpha)+\tilde{D}u(\alpha) \tag{4.7}$$

where $j=1,\dots,d$ and $\alpha,j\alpha$ are words in $\tilde{\Lambda}$, and

$$x:\tilde{\Lambda}\to\mathcal{H},\quad u:\tilde{\Lambda}\to\mathcal{U},\quad y:\tilde{\Lambda}\to\mathcal{Y}.$$

If $x(\emptyset)$ and u are known, then using $\sum_{U,\tilde{U}}$ we can compute x and y recursively. Such a $\tilde{\Lambda}$-linear system is also called a noncommutative Fornasini-Marchesini system in [BGM06] in reference to [FM78].

Let $z = (z_1, \ldots, z_d)$ be a d-tuple of formal noncommuting indeterminates. Define the Fourier transform of x, u and y as

$$\hat{x}(z) = \sum_{\alpha \in \tilde{\Lambda}} x(\alpha) z^\alpha, \quad \hat{u}(z) = \sum_{\alpha \in \tilde{\Lambda}} u(\alpha) z^\alpha, \quad \hat{y}(z) = \sum_{\alpha \in \tilde{\Lambda}} y(\alpha) z^\alpha$$

respectively where $z^\alpha = z_{\alpha_n} \ldots z_{\alpha_1}$ for $\alpha = \alpha_n \ldots \alpha_1 \in \tilde{\Lambda}$. Assuming that z-variables commute with the coefficients the input-output relation

$$\hat{y}(z) = \Theta_{U,\tilde{U}}(z) \hat{u}(z)$$

can be obtained on setting $x(\emptyset) := 0$ where

$$\Theta_{U,\tilde{U}}(z) := \sum_{\alpha \in \tilde{\Lambda}} \Theta^{(\alpha)}_{U,\tilde{U}} z^\alpha := \tilde{D} + \tilde{C} \sum_{\beta \in \tilde{\Lambda}, j=1,\ldots,d} (E_{\bar{\beta}})^* F_j^* z^{\beta j}. \tag{4.8}$$

Here $\bar{\beta} = \beta_1 \ldots \beta_n$ is the reverse of $\beta = \beta_n \ldots \beta_1 \in \tilde{\Lambda}$ and $\Theta^{(\alpha)}_{U,\tilde{U}}$ maps $\mathcal{U}$ to $\mathcal{Y}$. The formal noncommutative power series $\Theta_{U,\tilde{U}}$ is called the *transfer function* associated to the unitaries U and $\tilde{U}$. The transfer function is a mathematical tool for encoding the evolution of a $\tilde{\Lambda}$-linear system.

Theorem 4.1. *The map $M_{\Theta_{U,\tilde{U}}} : \ell^2(\tilde{\Lambda}, \mathcal{U}) \to \ell^2(\tilde{\Lambda}, \mathcal{Y})$ defined by*

$$M_{\Theta_{U,\tilde{U}}} \hat{u}(z) := \Theta_{U,\tilde{U}}(z) \hat{u}(z)$$

is a contraction.

Proof. Observe that $P_{\mathcal{Y}} \tilde{U}^* P_1 U(\tilde{h} \otimes \Omega^{\mathcal{K}}_\infty) = 0$ for all $\tilde{h} \in \tilde{\mathcal{H}}$. Consider another colligation which is defined as follows:

$$\mathcal{C}^\circ_{U,\tilde{U}} := \begin{pmatrix} E_1^{*\circ} & F_1^{*\circ} \\ \vdots & \vdots \\ E_d^{*\circ} & F_d^{*\circ} \\ \tilde{C}^\circ & \tilde{D} \end{pmatrix} : \mathcal{H}^\circ \oplus \mathcal{U} \to \bigoplus_{j=1}^d \mathcal{H}^\circ \oplus \mathcal{Y}$$

where $E_j^{*\circ} := P_{\mathcal{H}^\circ} E_j^*|_{\mathcal{H}^\circ} : \mathcal{H}^\circ \to \mathcal{H}^\circ$, $F_j^{*\circ} := P_{\mathcal{H}^\circ} F_j^* : \mathcal{U} \to \mathcal{H}^\circ$ and $\tilde{C}^\circ := \tilde{C}|_{\mathcal{H}^\circ} : \mathcal{H}^\circ \to \mathcal{Y}$ for $j = 1, \ldots, d$. Consider the outgoing Cuntz scattering system $((\mathcal{H} \otimes \mathcal{K}_\infty)^\circ, \underline{V}^E = (V_1^E, \ldots, V_d^E), \mathcal{G}_*^+, \mathcal{G})$, with $(\mathcal{H} \otimes \mathcal{K}_\infty)^\circ = \mathcal{H}^\circ \oplus \mathcal{G}$, constructed by us in Theorem 3.6. In Chapter 5.2 of [BV05] it is shown that there is an associated unitary colligation

$$\begin{pmatrix} \hat{E}_1 & \hat{F}_1 \\ \vdots & \vdots \\ \hat{E}_d & \hat{F}_d \\ \hat{M} & \hat{N} \end{pmatrix} : \mathcal{H}^\circ \oplus \mathcal{E} \to \bigoplus_{j=1}^d \mathcal{H}^\circ \oplus \mathcal{E}_* \tag{4.9}$$

such that $(\hat{E}_j, \hat{F}_j) = P_{\mathcal{H}^\circ} (V_j^E)^*|_{\mathcal{H}^\circ \oplus \mathcal{E}}$ and $(\hat{M}, \hat{N}) = P_{\mathcal{E}_*}|_{\mathcal{H}^\circ \oplus \mathcal{E}}$.
From equations (2.2) and (2.5) we observe that $(E_j^{*\circ}, F_j^{*\circ}) = P_{\mathcal{H}^\circ \otimes \epsilon_j} U|_{\mathcal{H}^\circ \oplus \mathcal{E}}$ (identifying $\mathcal{H}^\circ$

with $\mathcal{H}^\circ \otimes \epsilon_j$) and $(\tilde{C}^\circ, \tilde{D}) = P_{\mathcal{Y}} \tilde{U}^* P_1 U|_{\mathcal{H}^\circ \oplus \mathcal{E}}$. Using these observations we obtain the following relations:

$$\begin{aligned} U^*(E_j^{*\circ}, F_j^{*\circ}) &= U^* P_{\mathcal{H}^\circ \otimes \epsilon_j} U|_{\mathcal{H}^\circ \oplus \mathcal{E}} = P_{U^*(\mathcal{H}^\circ \otimes \epsilon_j)}|_{\mathcal{H}^\circ \oplus \mathcal{E}} = P_{V_j^E \mathcal{H}^\circ}|_{\mathcal{H}^\circ \oplus \mathcal{E}} \\ &= V_j^E P_{\mathcal{H}^\circ} (V_j^E)^*|_{\mathcal{H}^\circ \oplus \mathcal{E}} = V_j^E (\hat{E}_j, \hat{F}_j) \end{aligned} \tag{4.10}$$

for $j = 1, \ldots, d$ and

$$\begin{aligned} U^* \tilde{U}(\tilde{C}^\circ, \tilde{D}) &= U^* \tilde{U} P_{\mathcal{Y}} \tilde{U}^* P_1 U|_{\mathcal{H}^\circ \oplus \mathcal{E}} = U^* P_{\tilde{U}\mathcal{Y}} P_1 U|_{\mathcal{H}^\circ \oplus \mathcal{E}} = U^* P_{\tilde{U}\mathcal{Y}} U|_{\mathcal{H}^\circ \oplus \mathcal{E}} \\ &= P_{U^* \tilde{U} \mathcal{Y}}|_{\mathcal{H}^\circ \oplus \mathcal{E}} = P_{W^* \mathcal{Y}}|_{\mathcal{H}^\circ \oplus \mathcal{E}} \quad \text{(by equation (3.1))} \\ &= P_{\mathcal{E}_*}|_{\mathcal{H}^\circ \oplus \mathcal{E}} = (\hat{M}, \hat{N}). \end{aligned} \tag{4.11}$$

Let $\hat{u}(z) = \sum_{\alpha \in \tilde{\Lambda}} u(\alpha) z^\alpha \in \ell^2(\tilde{\Lambda}, \mathcal{U})$ with $u(\alpha) \in \mathcal{U}$ such that $\sum_{\alpha \in \tilde{\Lambda}} \|u(\alpha)\|^2 < \infty$. We would prove that

$$\|M_{\Theta_{U,\tilde{U}}} \hat{u}(z)\|^2 \leq \|\hat{u}(z)\|^2.$$

Define $x : \tilde{\Lambda} \to \mathcal{H}$ by equation (4.6) such that $x(\emptyset) = 0$. Further, define $x^\circ(\alpha) := P_{\mathcal{H}^\circ} x(\alpha)$ for all $\alpha \in \tilde{\Lambda}$. Now applying the projection $P_{\mathcal{H}^\circ}$ to relation (4.6) on both sides and using the fact $\tilde{\mathcal{H}}$ is invariant under E_j^* for $j = 1, \ldots, d$ we obtain the following relation:

$$x^\circ(j\alpha) = E_j^{*\circ} x^\circ(\alpha) + F_j^{*\circ} u(\alpha) \text{ for all } \alpha \in \tilde{\Lambda},\ j = 1, \ldots, d. \tag{4.12}$$

Because $P_{\mathcal{Y}} \tilde{U}^* P_1 U(\tilde{h} \otimes \Omega_\infty^{\mathcal{K}}) = 0$ for all $\tilde{h} \in \tilde{\mathcal{H}}$ we conclude by equation (4.5) that

$$\tilde{C}\tilde{h} = 0 \quad \text{for } \tilde{h} \in \tilde{\mathcal{H}}. \tag{4.13}$$

This implies

$$\tilde{C}x(\alpha) = \tilde{C}^\circ x^\circ(\alpha) \text{ for all } \alpha \in \tilde{\Lambda}. \tag{4.14}$$

Define $y : \tilde{\Lambda} \to \mathcal{Y}$ by

$$y(\alpha) := \tilde{C}x(\alpha) + \tilde{D}u(\alpha) \tag{4.15}$$

for all $\alpha \in \tilde{\Lambda}$. Recall that the input-output relation stated just before the theorem is

$$\hat{y}(z) = \sum_{\alpha \in \tilde{\Lambda}} y(\alpha) z^\alpha = \Theta_{U,\tilde{U}}(z)\hat{u}(z) (= M_{\Theta_{U,\tilde{U}}} \hat{u}(z)).$$

Using the unitary colligation given in equation (4.9) we have

$$\begin{aligned} \|x^\circ(\alpha)\|^2 + \|u(\alpha)\|^2 &= \sum_{j=1}^d \|\hat{E}_j x^\circ(\alpha) + \hat{F}_j u(\alpha)\|^2 + \|\hat{M} x^\circ(\alpha) + \hat{N} u(\alpha)\|^2 \\ &= \sum_{j=1}^d \|E_j^{*\circ} x^\circ(\alpha) + F_j^{*\circ} u(\alpha)\|^2 + \|\tilde{C}^\circ x^\circ(\alpha) + \tilde{D} u(\alpha)\|^2 \\ &= \sum_{j=1}^d \|x^\circ(j\alpha)\|^2 + \|\tilde{C}x(\alpha) + \tilde{D}u(\alpha)\|^2 \\ &= \sum_{j=1}^d \|x^\circ(j\alpha)\|^2 + \|y(\alpha)\|^2 \end{aligned}$$

for all $\alpha \in \tilde{\Lambda}$. In the above calculation equations (4.10), (4.11), (4.12), (4.14) and (4.15) respectively have been used. This gives us

$$\|u(\alpha)\|^2 - \|y(\alpha)\|^2 = \sum_{j=1}^{d} \|x^\circ(j\alpha)\|^2 - \|x^\circ(\alpha)\|^2$$

for all $\alpha \in \tilde{\Lambda}$. Summing over all $\alpha \in \tilde{\Lambda}$ with $|\alpha| \leq n$ and using the fact that $x^\circ(\emptyset) = 0$ we obtain

$$\sum_{|\alpha|\leq n} \|u(\alpha)\|^2 - \sum_{|\alpha|\leq n} \|y(\alpha)\|^2 = \sum_{|\alpha|=n+1} \|x^\circ(\alpha)\|^2 \geq 0 \text{ for all } n \in \mathbb{N}.$$

Therefore

$$\sum_{|\alpha|\leq n} \|y(\alpha)\|^2 \leq \sum_{|\alpha|\leq n} \|u(\alpha)\|^2 \text{ for all } n \in \mathbb{N}.$$

Finally taking limit $n \to \infty$ both the sides we get that $M_{\Theta_{U,\tilde{U}}}$ is a contraction. □

$M_{\Theta_{U,\tilde{U}}}$ is a *multi-analytic operators* ([Po95]) (also called *analytic intertwining operator* in [BV05]) because

$$M_{\Theta_{U,\tilde{U}}}(\sum_{\alpha\in\tilde{\Lambda}} u(\alpha)z^\alpha z^j) = M_{\Theta_{U,\tilde{U}}}(\sum_{\alpha\in\tilde{\Lambda}} u(\alpha)z^\alpha)z^j \quad \text{for } j = 1, \ldots, d,$$

i.e., $M_{\Theta_{U,\tilde{U}}}$ intertwines with right translation. The noncommutative power series $\Theta_{U,\tilde{U}}$ is called the *symbol* of $M_{\Theta_{U,\tilde{U}}}$.

5 Transfer Functions, Observability and Scattering

We would now establish that the transfer function can be derived from the coisometry W of section 3. In the last section d-tuple $z = (z_1, \ldots, z_d)$ of formal noncommuting indeterminates were employed. Treat $(z^\alpha)_{\alpha\in\tilde{\Lambda}}$ as an orthonormal basis of $\ell^2(\tilde{\Lambda}, \mathbb{C})$. Assume $\mathcal{Y}$ and $\mathcal{U}$ to be the spaces associated with our model with unitaries U and $\tilde{U}$ as in the last section. For $y(\alpha) \in \mathcal{Y}$ with $\sum_{\alpha\in\tilde{\Lambda}} \|y(\alpha)\|^2 < \infty$, any series $\sum_{\alpha\in\tilde{\Lambda}} y(\alpha)z^\alpha$ stands for a series converging to an element of $\ell^2(\tilde{\Lambda}, \mathcal{Y})$. It follows from Remark 3.7 that there exist a unitary operator $\tilde{\Gamma} : (\tilde{\mathcal{H}} \otimes \mathcal{K}_\infty)^\circ \to \ell^2(\tilde{\Lambda}, \mathcal{Y})$ defined by

$$\tilde{\Gamma}(V_\alpha^C y) := yz^{\bar{\alpha}} \quad \text{for all} \quad \alpha \in \tilde{\Lambda}, y \in \mathcal{Y}.$$

We observe the following intertwining relation:

$$\tilde{\Gamma}(V_\alpha^C y) = (\tilde{\Gamma}y)z^{\bar{\alpha}}. \tag{5.1}$$

Similarly, using Theorem 3.6, we can define a unitary operator $\Gamma : (\mathcal{H} \otimes \mathcal{K}_\infty)^\circ (= (\mathcal{H}^\circ \oplus \mathcal{G})) \to \mathcal{H}^\circ \oplus \ell^2(\tilde{\Lambda}, \mathcal{U})$ by

$$\Gamma(\mathring{h} \oplus V_\alpha^E \eta) := \mathring{h} \oplus \eta z^{\bar{\alpha}} \text{ for all} \quad \alpha \in \tilde{\Lambda}$$

where $\mathring{h} \in \mathcal{H}^\circ, \eta \in \mathcal{U}$. In this case the intertwining relation is

$$\Gamma(V_\alpha^E \eta) = (\Gamma\eta)z^{\bar{\alpha}}. \tag{5.2}$$

Using the coisometric operator W, which appears in Remark 2.3, we define Γ_W by the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{H}\otimes\mathcal{K}_\infty)^\circ & \xrightarrow{W} & (\tilde{\mathcal{H}}\otimes\mathcal{K}_\infty)^\circ \\ \Gamma\big\downarrow & & \big\downarrow\tilde{\Gamma} \\ \mathcal{H}^\circ\oplus\ell^2(\tilde{\Lambda},\mathcal{U}) & \xrightarrow{\Gamma_W} & \ell^2(\tilde{\Lambda},\mathcal{Y}), \end{array} \tag{5.3}$$

i.e., $\Gamma_W = \tilde{\Gamma}W\Gamma^{-1}$.

Theorem 5.1. Γ_W *defined by the above commutative diagram satisfies*

$$\Gamma_W|_{\ell^2(\tilde{\Lambda},\,\mathcal{U})} = M_{\Theta_{U,\tilde{U}}}.$$

Proof. Using the intertwining relation $V_j^C W = WV_j^E$ from Remark 2.3, and equations (5.1) and (5.2) we obtain

$$\begin{aligned}\Gamma_W(\eta z^\beta z^j) &= \tilde{\Gamma}W\Gamma^{-1}(\eta z^\beta z^j) = \tilde{\Gamma}WV_j^E V_{\bar{\beta}}^E\eta \\ &= \tilde{\Gamma}V_j^C V_{\bar{\beta}}^C W\eta = (\tilde{\Gamma}W\eta)z^\beta z^j = \Gamma_W(\eta z^\beta)z^j\end{aligned}$$

for $\eta\in\mathcal{U},\beta\in\tilde{\Lambda},j=1,\dots,d$. Hence, $\Gamma_W|_{\ell^2(\tilde{\Lambda},\,\mathcal{U})}$ is a multi-analytic operator. For computing its symbol we determine $\Gamma_W\eta$ for $\eta\in\mathcal{U}$, where η is identified with $\eta z^\phi\in\ell^2(\tilde{\Lambda},\mathcal{U})$. For $\alpha=\alpha_{n-1}\dots\alpha_1\in\tilde{\Lambda}$ let P_α be the orthogonal projection onto

$$\begin{aligned}&\tilde{\Gamma}^{-1}\{f\in\ell^2(\tilde{\Lambda},\mathcal{Y}) : f = yz^\alpha \text{ for some } y\in Y\} \\ =\ & V_{\bar{\alpha}}^C\mathcal{Y} = \tilde{U}_1^*\dots\tilde{U}_{n-1}^*(\tilde{\mathcal{H}}\otimes\epsilon_{\alpha_1}\otimes\cdots\otimes\epsilon_{\alpha_{n-1}}\otimes(\Omega_n^{\mathcal{K}})^\perp\otimes\Omega_{[n+1,\infty)}^{\mathcal{K}})\end{aligned}$$

with $\tilde{U}_i$'s as in Proposition 2.1.

Recall that the tuple $\underline{E}$ associated with the unitary U is a lifting of the tuple $\underline{C}$ (associated with the unitary $\tilde{U}$) and so $\underline{E}$ can be written as a block matrix in terms of $\underline{C}$ as follows: $E_j = \begin{pmatrix} C_j & 0 \\ B_j & A_j\end{pmatrix}$ for $j=1,\dots,d$ w.r.t. to the decomposition $\mathcal{H}=\tilde{\mathcal{H}}\oplus\mathcal{H}^\circ$ where $\underline{B}$ and $\underline{A}$ are some row contractions. Because $\underline{E}$ is a coisometric lifting of $\underline{C}$ we have

$$\sum_{j=1}^d C_jC_j^* = I \text{ and } \sum_{j=1}^d C_jB_j^* = 0$$

(cf. [DG11]) . Now using these relations and equations (4.2), (4.3) and (4.4) it can be easily verified that

$$P_\alpha\tilde{U}_1^*\dots\tilde{U}_n^*P_nU_n\dots U_1\eta = P_\alpha\tilde{U}_1^*\dots\tilde{U}_m^*P_mU_m\dots U_1\eta \text{ for all } m\ge n,\eta\in\mathcal{U}.$$

Using the formula of W from Proposition 2.1 we obtain

$$P_\alpha W\eta = P_\alpha\tilde{U}_1^*\dots\tilde{U}_n^*P_nU_n\dots U_1\eta \text{ for } \eta\in\mathcal{U}.$$

Finally for $\eta\in\mathcal{U}$

$$P_\alpha\tilde{U}_1^*\dots\tilde{U}_n^*P_nU_n\dots U_1\eta = \begin{cases}\tilde{D}\eta & \text{if } n=1,\alpha=\emptyset, \\ V_{\bar{\alpha}}^C(\tilde{C}E_{\alpha_{n-1}}^*\dots E_{\alpha_2}^*F_{\alpha_1}^*\eta) & \text{if } n=|\alpha|+1\ge 2.\end{cases}$$

This implies for $\eta \in \mathcal{U}$

$$\tilde{\Gamma} W \Gamma^{-1} \eta = \tilde{\Gamma} W \eta = \tilde{D}\eta \oplus \sum_{|\alpha| \geq 1} (\tilde{C} E^*_{\alpha_{n-1}} \dots E^*_{\alpha_2} F^*_{\alpha_1} \eta) z^\alpha.$$

Comparing this with equation (4.8) we conclude that $\Gamma_W|_{\ell^2(\tilde{\Lambda},\, \mathcal{U})} = M_{\Theta_{U,\tilde{U}}}$. □

We illustrate below that the notion of observability from the linear systems theory is closely related to the scattering theory notions for our model and the outgoing Cuntz scattering system.

Definition 5.2. *The observability operator $W_0 : \mathcal{H}^\circ \to \ell^2(\tilde{\Lambda}, \mathcal{Y})$ is defined as the restriction of the operator Γ_W to $\mathcal{H}^\circ$, i.e., $W_0 = \Gamma_W|_{\mathcal{H}^\circ}$.*

It follows that $W_0 \mathring{h} = (\tilde{C}(E_{\bar{\alpha}})^* \mathring{h})_{\alpha \in \tilde{\Lambda}}$. Popescu has studied the similar types of operators called Poisson kernels in [Po99].

Definition 5.3. *If there exist $k, K > 0$ such that for all $\mathring{h} \in \mathcal{H}^\circ$*

$$k\|\mathring{h}\|^2 \leq \sum_{\alpha \in \tilde{\Lambda}} \|\tilde{C}(E_{\bar{\alpha}})^* \mathring{h}\|^2 = \|W_0 \mathring{h}\|^2 \leq K\|\mathring{h}\|^2,$$

then the $\tilde{\Lambda}-$linear system is called (uniformly) observable.

Observability of a system for $\dim \mathcal{H} < \infty$ is interpreted as the property of the system that in the absence of $\mathcal{U}$-inputs we can determine the original state $h \in \mathcal{H}^\circ$ of the system from all $\mathcal{Y}$-outputs at all times. Uniform observability is an analog of this for $\dim \mathcal{H} = \infty$.

We extend W_0 to

$$\widehat{W}_0 : (\tilde{\mathcal{H}} \oplus \mathcal{H}^\circ)(= \mathcal{H}) \longrightarrow \tilde{\mathcal{H}} \oplus \ell^2(\tilde{\Lambda}, \mathcal{Y})$$

by defining $\widehat{W}_0 \tilde{h} := \tilde{h}$ for all $\tilde{h} \in \tilde{\mathcal{H}}$. If W_0 is uniformly observable, then using $\hat{k} = k$ and $\hat{K} = \max\{1, K\}$ the above inequalities can be extended to $\widehat{W}_0$ on $\mathcal{H}$ as

$$\hat{k}\|h\|^2 \leq \|\widehat{W}_0 h\|^2 \leq \hat{K}\|h\|^2$$

for all $h \in \mathcal{H}$.

Before stating the main theorem of this section regarding observability we recall from [DG11] the following: Let $\underline{C}$ be a row contraction on a Hilbert space $\mathcal{H}_C$. The lifting $\underline{E}$ of $\underline{C}$ is called *subisometric* [DG11] if the minimal isometric dilations $\underline{\widehat{V}}^E$ and $\underline{\widehat{V}}^C$ of $\underline{E}$ and $\underline{C}$ respectively are unitarily equivalent and the corresponding unitary, which intertwines between $\widehat{V}_i^E$ and $\widehat{V}_i^C$ for all $i = 1, 2, \dots, d$, acts as identity on $\mathcal{H}_C$.

Theorem 5.4. *For any $\tilde{\Lambda}$-linear system associated to a generalized repeated interaction model with unitaries $U, \tilde{U}$ the following statements are equivalent:*

(a) The system is (uniformly) observable.

(b) The observability operator W_0 is isometric.

(c) The tuple $\underline{E}$ associated with the unitary U is a subisometric lifting of the tuple $\underline{C}$ (associated with the unitary $\tilde{U}$).

(d) $W : (\mathcal{H} \otimes \mathcal{K}_\infty)^\circ \to (\tilde{\mathcal{H}} \otimes \mathcal{K}_\infty)^\circ$ is unitary.

If one of the above holds, then

(e) *The transfer function* $\Theta_{U,\tilde{U}}$ *is inner, i.e.,* $M_{\Theta_{U,\tilde{U}}} : \ell^2(\tilde{\Lambda}, \mathcal{U}) \to \ell^2(\tilde{\Lambda}, \mathcal{Y})$ *is isometric.*

If we have additional assumptions, viz. $\dim \mathcal{H} < \infty$ *and* $\dim \mathcal{P} \geq 2$*, then the converse holds, i.e.,* (e) *implies all of* $(a), (b), (c)$ *and* (d)*.*

Proof. Clearly $(d) \Rightarrow (b) \Rightarrow (a)$. We now prove $(a) \Rightarrow (d)$. Because the system is (uniformly) observable there exist $k > 0$ such that for all $\mathring{h} \in \mathcal{H}^\circ$

$$k\|\mathring{h}\|^2 \leq \|W_0 \mathring{h}\|^2.$$

Since $\bigcup_{m\geq 1} \tilde{\mathcal{H}} \otimes \mathcal{K}_{[1,m]}$ is a dense subspace of $\mathcal{H} \otimes \mathcal{K}_\infty$, for any $0 \neq \eta \in \mathcal{H} \otimes \mathcal{K}_\infty$ there exist $n \in \mathbb{N}$ and $\eta' \in \mathcal{H} \otimes \mathcal{K}_{[1,n]}$ such that

$$\|\eta - \eta'\| < \frac{\sqrt{k}}{\sqrt{k}+1}\|\eta\|.$$

Let $\eta_0 \in \mathcal{H} \otimes \mathcal{K}_{[1,n]}$. Suppose $U_n \dots U_1 \eta_0 = h_0 \otimes p_0 \otimes \Omega^{\mathcal{K}}_{[n+1,\infty)}$, where $h_0 \in \mathcal{H}$, $p_0 \in \mathcal{P}_{[1,n]}$. Then clearly

$$\lim_{N\to\infty} \|\tilde{U}_1^* \dots \tilde{U}_n^* \tilde{U}_{n+1}^* \dots \tilde{U}_N^* P_N U_N \dots U_{n+1} U_n \dots U_1 \eta_0\| = \|\widehat{W}_0 h_0\| \|p_0\|$$

and thus by Proposition 2.1 it is equal to $\|\widehat{W}\eta_0\|$. Because the system is (uniformly) observable,

$$\|\widehat{W}_0 h_0\| \|p_0\| \geq \sqrt{k}\|h_0\|\|p_0\|.$$

Therefore $\|\widehat{W}\eta_0\|^2 \geq k\|\eta_0\|^2$. However, in general $U_n \dots U_1 \eta_0 = \sum_j h_0^{(j)} \otimes p_0^{(j)} \otimes \Omega^{\mathcal{K}}_{[n+1,\infty)}$ with $h_0^{(j)} \in \mathcal{H}$ and some mutually orthogonal vectors $p_0^{(j)} \in \mathcal{P}_{[1,n]}$. By using the above inequality for each term of the summation and then adding them we find that in general for all $\eta_0 \in \mathcal{H} \otimes \mathcal{K}_{[1,n]}$

$$\|\widehat{W}\eta_0\|^2 \geq k\|\eta_0\|^2.$$

In particular, for $\eta' \in \mathcal{H} \otimes \mathcal{K}_{[1,n]}$ we have the above inequality. Therefore

$$\begin{aligned} \|\widehat{W}\eta\| &\geq \|\widehat{W}\eta'\| - \|\widehat{W}(\eta' - \eta)\| \\ &\geq \sqrt{k}\|\eta'\| - \|\eta - \eta'\| \\ &\geq \sqrt{k}\|\eta\| - (\sqrt{k}+1)\|\eta - \eta'\| > 0. \end{aligned}$$

This implies $\widehat{W}\eta \neq 0$ for all $0 \neq \eta \in \mathcal{H} \otimes \mathcal{K}_\infty$ and hence $\widehat{W}$ is injective. Recall that $\widehat{W}$ is a coisometry and an injective coisometry is unitary. Further, because $\widehat{W}(\tilde{h} \otimes \Omega^{\mathcal{K}}_\infty) = \tilde{h} \otimes \Omega^{\mathcal{K}}_\infty$ for all $\tilde{h} \in \tilde{\mathcal{H}}$ it follows that W is unitary. This establishes $(a) \Rightarrow (d)$ and we have proved $(a) \Leftrightarrow (b) \Leftrightarrow (d)$.

Next we prove $(d) \Leftrightarrow (c)$. Assume that (d) holds. Since W is unitary, clearly $\widehat{W}$ is unitary. We know that $\widehat{W}$ intertwines between the minimal isometric dilations $\underline{\widehat{V}}^E$ and $\underline{\widehat{V}}^C$ of $\underline{E}$ and $\underline{C}$ respectively. Hence $\underline{E}$ is a subisometric lifting of $\underline{C}$.

Conversely, if we assume (c), then by the definition of subisometric lifting there exist a unitary operator

$$\widehat{W}_1 : \mathcal{H} \otimes \mathcal{K}_\infty \longrightarrow \tilde{\mathcal{H}} \otimes \mathcal{K}_\infty$$

which intertwines between $\underline{\widehat{V}}^E$ and $\underline{\widehat{V}}^C$, and $\widehat{W}_1$ acts as an identity on $\tilde{\mathcal{H}} \otimes \Omega^{\mathcal{K}}_\infty$. To prove W is unitary it is enough to prove $\widehat{W}$ is unitary. We show that $\widehat{W} = \widehat{W}_1$. By the definition of the

minimal isometric dilation we know that $\tilde{\mathcal{H}} \otimes \mathcal{K}_\infty = \overline{span}\{\widehat{V}^C_\alpha(\tilde{h} \otimes \Omega^{\mathcal{K}}_\infty) : \tilde{h} \in \tilde{\mathcal{H}}, \alpha \in \tilde{\Lambda}\}$. For $\tilde{h} \in \tilde{\mathcal{H}}$, by equation (2.1) and Proposition 2.2,

$$\begin{aligned}\widehat{W}^*\widehat{V}^C_j(\tilde{h} \otimes \Omega^{\mathcal{K}}_\infty) &= \widehat{V}^E_j\widehat{W}^*(\tilde{h} \otimes \Omega^{\mathcal{K}}_\infty) = \widehat{V}^E_j(\tilde{h} \otimes \Omega^{\mathcal{K}}_\infty) \\ &= \widehat{W}_1^*\widehat{V}^C_j\widehat{W}_1(\tilde{h} \otimes \Omega^{\mathcal{K}}_\infty) = \widehat{W}_1^*\widehat{V}^C_j(\tilde{h} \otimes \Omega^{\mathcal{K}}_\infty).\end{aligned}$$

Thus $\widehat{W}^* = \widehat{W}_1^*$ and hence $\widehat{W} = \widehat{W}_1$.

To prove $(d) \Rightarrow (e)$ we at first note that since W is unitary, Γ_W is also unitary. By Theorem 4.2, we have $M_{\Theta_{U,\tilde{U}}} = \Gamma_W|_{\ell^2(\tilde{\Lambda},\mathcal{U})}$. Since a restriction of a unitary operator is an isometry, $M_{\Theta_{U,\tilde{U}}}$ is isometric.

Now using the additional assumptions $\dim \mathcal{H} < \infty$ and $\dim \mathcal{P} \geq 2$, we show $(e) \Rightarrow (b)$. Define

$$\mathcal{H}_{scat} := \mathcal{H} \cap \widehat{W}^*(\tilde{\mathcal{H}} \otimes \mathcal{K}_\infty) = \tilde{\mathcal{H}} \oplus \{\mathring{h} \in \mathcal{H}^\circ : \|W_0\mathring{h}\| = \|\mathring{h}\|\}.$$

Since $\|\widehat{W}_0 h\| = \lim_{n\to\infty} \|\tilde{U}_1 \dots \tilde{U}_n\tilde{P}_nU_n \dots U_1h\|$ by Proposition 2.1, the following can be easily verified:

$$U(\mathcal{H}_{scat} \otimes \Omega^{\mathcal{K}}) \subset \mathcal{H}_{scat} \otimes \mathcal{P}. \tag{5.4}$$

Because $M_{\Theta_{U,\tilde{U}}} = \Gamma_W|_{\ell^2(\tilde{\Lambda},\mathcal{U})}$ is isometric by (e), it can be checked that

$$U(\mathcal{H} \otimes (\Omega^{\mathcal{K}})^\perp) \subset \mathcal{H}_{scat} \otimes \mathcal{P}. \tag{5.5}$$

Combining equations (5.4) and (5.5) we have

$$U^*((\mathcal{H} \ominus \mathcal{H}_{scat}) \otimes \mathcal{P}) \subset (\mathcal{H} \ominus \mathcal{H}_{scat}) \otimes \Omega^{\mathcal{K}}. \tag{5.6}$$

Since $\dim \mathcal{H} < \infty$ and $\dim \mathcal{P} \geq 2$, comparing the dimensions of the both the sides of equation (5.6), we obtain $\mathcal{H} \ominus \mathcal{H}_{scat} = \{0\}$, i.e., $\mathcal{H} = \mathcal{H}_{scat}$. This implies W_0 is isometric and hence $(e) \Rightarrow (b)$. □

6 Transfer Functions and Characteristic Functions of Liftings

Continuing with the study of our generalized repeated interaction model, from the equation (4.4), we have

$$\tilde{U}^*(\tilde{h} \otimes \epsilon_j) = ((C_j\tilde{h} \otimes \Omega^{\mathcal{K}}) \oplus D_j\tilde{h}) \quad \text{for } \tilde{h} \in \tilde{\mathcal{H}} \text{ and } j = 1, \dots, d.$$

Moreover, because $\widehat{V}^C_j(\tilde{h} \otimes \Omega^{\mathcal{K}}_\infty) = \tilde{U}^*(\tilde{h} \otimes \epsilon_j) \otimes \Omega^{\mathcal{K}}_{[2,\infty)}$, we obtain

$$\widehat{V}^C_j(\tilde{h} \otimes \Omega^{\mathcal{K}}_\infty) = ((C_j\tilde{h} \otimes \Omega^{\mathcal{K}}_1) \oplus D_j\tilde{h}) \otimes \Omega^{\mathcal{K}}_{[2,\infty)} \quad \text{for } \tilde{h} \in \tilde{\mathcal{H}} \text{ and } j = 1, \dots, d. \tag{6.1}$$

Let $D_C := (I - \underline{C}^*\underline{C})^{\frac{1}{2}} : \bigoplus_{i=1}^d \tilde{\mathcal{H}} \to \bigoplus_{i=1}^d \tilde{\mathcal{H}}$ denote the defect operator and $\mathcal{D}_C := \overline{\text{Range } D_C}$. The full Fock space over $\mathbb{C}^d$ ($d \geq 2$) denoted by $\mathcal{F}$ is

$$\mathcal{F} = \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d)^{\otimes^2} \oplus \dots \oplus (\mathbb{C}^d)^{\otimes^m} \oplus \cdots.$$

The vector $e_\emptyset := 1 \oplus 0 \oplus \cdots$ is called the vacuum vector. Let $\{e_1, \dots, e_d\}$ be the standard orthonormal basis of $\mathbb{C}^d$. For $\alpha \in \tilde{\Lambda}$ and $|\alpha| = n$, e_α denote the vector $e_{\alpha_1} \otimes e_{\alpha_2} \otimes \dots \otimes e_{\alpha_n}$ in

the full Fock space $\mathcal{F}$. We recall that Popescu's construction [Po89a] of the minimal isometric dilation $\underline{\tilde{V}}^C = (\tilde{V}_1^C, \ldots, \tilde{V}_d^C)$ on $\tilde{\mathcal{H}} \oplus (\mathcal{F} \otimes \mathcal{D}_C)$ of the tuple $\underline{C}$ is

$$\tilde{V}_j^C(\tilde{h} \oplus \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha) = C_j\tilde{h} \oplus [e_\emptyset \otimes (D_C)_j\tilde{h} + e_j \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha]$$

for $\tilde{h} \in \tilde{\mathcal{H}}$ and $d_\alpha \in \mathcal{D}_C$ where $(D_C)_j\tilde{h} = D_C(0, \ldots, \tilde{h}, \ldots, 0)$ ($\tilde{h}$ is embedded at the j^{th} component). So

$$\tilde{V}_j^C\tilde{h} = C_j\tilde{h} \oplus (e_\emptyset \otimes (D_C)_j\tilde{h}) \quad \text{for } \tilde{h} \in \tilde{\mathcal{H}} \text{ and } j = 1, \ldots, d. \tag{6.2}$$

From equations (6.1) and (6.2) it follows that

$$\| \sum_{j=1}^d D_j\tilde{h}_j\|^2 = \| \sum_{j=1}^d (D_C)_j\tilde{h}_j\|^2 \tag{6.3}$$

where $\tilde{h}_j \in \tilde{\mathcal{H}}$ for $j = 1, \ldots, d$. Let $\Phi_C : \overline{span}\{D_j\tilde{h} : \tilde{h} \in \tilde{\mathcal{H}}, j = 1, \ldots, d\} \to \mathcal{D}_C$ be the unitary given by

$$\Phi_C(\sum_{j=1}^d D_j\tilde{h}_j) = \sum_{j=1}^d (D_C)_j\tilde{h}_j \quad \text{for } \tilde{h}_j \in \tilde{\mathcal{H}} \text{ and } j = 1, \ldots, d.$$

Similarly with $D_E := (I - \underline{E}^*\underline{E})^{\frac{1}{2}} : \bigoplus_{i=1}^d \mathcal{H} \to \bigoplus_{i=1}^d \mathcal{H}$ and $\mathcal{D}_E := \overline{\text{Range } D_E}$ we can define another unitary operator $\Phi_E : \overline{span}\{F_jh : h \in \mathcal{H}, j = 1, \ldots, d\} \to \mathcal{D}_E$ by

$$\Phi_E(\sum_{j=1}^d F_jh_j) = \sum_{j=1}^d (D_E)_jh_j \quad \text{for } h_j \in \mathcal{H} \text{ and } j = 1, \ldots, d.$$

The second equation of (4.1) yields

$$\sum_{j=1}^d D_jD_j^*y = y \quad \text{for} \quad y \in \mathcal{Y}.$$

This implies

$$\overline{span}\{D_j\tilde{h} : \tilde{h} \in \tilde{\mathcal{H}}, j = 1, \ldots, d\} = \mathcal{Y}.$$

Similarly, we can show that $\overline{span}\{F_jh : h \in \mathcal{H}, j = 1, \ldots, d\} = \mathcal{U}$. Thus Φ_C is a unitary from $\mathcal{Y}$ onto $\mathcal{D}_C$ and Φ_E is a unitary from $\mathcal{U}$ onto $\mathcal{D}_E$. As a consequence we have

$$D_j^*D_i = (D_C)_j^*(D_C)_i = \delta_{ij}I - C_j^*C_i, \tag{6.4}$$
$$F_j^*F_i = (D_E)_j^*(D_E)_i = \delta_{ij}I - E_j^*E_i. \tag{6.5}$$

Define unitaries $\tilde{M}_{\Phi_C} : \ell^2(\tilde{\Lambda}, \mathcal{Y}) \to \mathcal{F} \otimes \mathcal{D}_C$ and $\tilde{\Phi}_E : \mathcal{U}z^\emptyset \to e_\emptyset \otimes \mathcal{D}_E$ by

$$\begin{aligned} \tilde{M}_{\Phi_C}\big(\sum_{\alpha \in \tilde{\Lambda}} y_\alpha z^\alpha\big) &:= \sum_{\alpha \in \tilde{\Lambda}} e_{\bar{\alpha}} \otimes \Phi_C(y_\alpha), \\ \tilde{\Phi}_E(uz^\emptyset) &:= e_\emptyset \otimes \Phi_E u \end{aligned}$$

which would be useful in comparing transfer functions with characteristic functions.

Define $D_{*,A} := (I - \underline{A}\underline{A}^*)^{\frac{1}{2}} : \mathcal{H}^\circ \to \mathcal{H}^\circ$ and $\mathcal{D}_{*,A} := \overline{\text{Range } D_{*,A}}$. Because $\underline{E}$ is a coisometric lifting of $\underline{C}$, using Theorem 2.1 of [DG11] we conclude that there exist an isometry $\gamma : \mathcal{D}_{*,A} \to \mathcal{D}_C$ with $\gamma D_{*,A} h = \underline{B}^* h$ for all $h \in \mathcal{H}^\circ$. Further, for $h \in \mathcal{H}^\circ$

$$\begin{aligned}
\Phi_C \tilde{C} h &= \Phi_C \sum_{j=1}^{d} D_j P_{\tilde{\mathcal{H}}} E_j^* h = \Phi_C \sum_{j=1}^{d} D_j P_{\tilde{\mathcal{H}}} (B_j^* h \oplus A_j^* h) \\
&= \Phi_C \sum_{j=1}^{d} D_j B_j^* h = \sum_{j=1}^{d} (D_C)_j B_j^* h \\
&= D_C \underline{B}^* h = \underline{B}^* h.
\end{aligned}$$

The last equality holds because for the coisometric tuple $\underline{C}$ the operator D_C is the projection onto $\mathcal{D}_C$ and Range $\underline{B}^* \subset \mathcal{D}_C$. This implies

$$\Phi_C \tilde{C} h = \gamma D_{*,A} h. \tag{6.6}$$

The characteristic function of lifting $\underline{E}$ of $\underline{C}$, which was introduced in [DG11], has the following expansion: For $h \in \tilde{\mathcal{H}}$

$$\Theta_{C,E} (D_E)_i h = e_\emptyset \otimes [(D_C)_i h - \gamma D_{*,A} B_i h] - \sum_{|\alpha| \geq 1} e_\alpha \otimes \gamma D_{*,A} (A_\alpha)^* B_i h \tag{6.7}$$

and for $h \in \mathcal{H}^\circ$

$$\Theta_{C,E} \, (D_E)_i h = -e_\emptyset \otimes \gamma D_{*,A} A_i h + \sum_{j=1}^{d} e_j \otimes \sum_{\alpha} e_\alpha \otimes \gamma D_{*,A} (A_\alpha)^* (\delta_{ji} I - A_j^* A_i) h. \tag{6.8}$$

Theorem 6.1. *Let U and $\tilde{U}$ be unitaries associated with a generalized repeated interaction model, and the lifting $\underline{E}$ of $\underline{C}$ be the corresponding lifting. Then the characteristic function $\Theta_{C,E}$ coincides with the transfer function $\Theta_{U,\tilde{U}}$, i.e.,*

$$\tilde{M}_{\Phi_C} \Theta_{U,\tilde{U}}(z) = \Theta_{C,E} \tilde{\Phi}_E.$$

Proof. If $h \in \mathcal{H}$ and $i = 1, \ldots, d$, then by equation (4.8)

$$\begin{aligned}
\tilde{M}_{\Phi_C} \Theta_{U,\tilde{U}}(z)(F_i h z^\emptyset) &= \tilde{M}_{\Phi_C} [\tilde{D} \, z^\emptyset + \sum_{\beta \in \tilde{\Lambda}, j=1,\ldots,d} \tilde{C} (E_{\bar{\beta}})^* F_j^* z^{\beta j}](F_i h z^\emptyset) \\
&= \tilde{M}_{\Phi_C} [\tilde{D} F_i h \, z^\emptyset + \sum_{\beta \in \tilde{\Lambda}, j=1,\ldots,d} \tilde{C} (E_{\bar{\beta}})^* F_j^* F_i h \, z^{\beta j}].
\end{aligned} \tag{6.9}$$

Case 1. $h \in \tilde{\mathcal{H}}$:

$$\begin{aligned}
\tilde{D} F_i h &= \sum_{j=1}^{d} D_j P_{\tilde{\mathcal{H}}} F_j^* F_i h = \sum_{j=1}^{d} D_j P_{\tilde{\mathcal{H}}} (\delta_{ij} I - E_j^* E_i) h \\
&= D_i h - \big(\sum_{j=1}^{d} D_j P_{\tilde{\mathcal{H}}} E_j^* \big) E_i h = D_i h - \tilde{C} E_i h \\
&= D_i h - \tilde{C} (C_i h \oplus B_i h) = D_i h - \tilde{C} B_i h.
\end{aligned}$$

Second and last equalities follows from equations (6.5) and (4.13) respectively. By equation (6.5) again we obtain

$$\begin{aligned}
\sum_{\beta\in\tilde{\Lambda},j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^* F_j^* F_i h \; z^{\beta j} &= \sum_{\beta\in\tilde{\Lambda},j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^*(\delta_{ij} I - E_j^* E_i) h \; z^{\beta j} \\
&= \sum_{\beta\in\tilde{\Lambda}} \tilde{C}(E_{\bar{\beta}})^* h \; z^{\beta i} - \sum_{\beta\in\tilde{\Lambda},j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^* E_j^* E_i h \; z^{\beta j} \\
&= - \sum_{\beta\in\tilde{\Lambda},j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^* E_j^* E_i h \; z^{\beta j} \\
&\quad \text{(because } \tilde{C}(E_{\bar{\beta}})^* h = \tilde{C}(C_{\bar{\beta}})^* h = 0 \text{ by equation (4.13))} \\
&= - \sum_{\beta\in\tilde{\Lambda},j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^* \big((C_j^* C_i + B_j^* B_i) h \oplus A_j^* B_i h\big) \; z^{\beta j} \\
&= - \sum_{\beta\in\tilde{\Lambda},j=1,\ldots,d} \tilde{C}(A_{\bar{\beta}})^* A_j^* B_i h \; z^{\beta j} \quad \text{(by equation (4.13))} \\
&= - \sum_{|\alpha|\geq 1} \tilde{C}(A_{\bar{\alpha}})^* B_i h \; z^{\alpha}.
\end{aligned}$$

So by equation (6.9) we have for all $i = 1, \ldots, d$ and $h \in \tilde{\mathcal{H}}$

$$\begin{aligned}
\tilde{M}_{\Phi_C} \Theta_{U,\tilde{U}}(z)(F_i h z^{\emptyset}) &= \tilde{M}_{\Phi_C}[(D_i h - \tilde{C} B_i h) \; z^{\emptyset} - \sum_{|\alpha|\geq 1} \tilde{C}(A_{\bar{\alpha}})^* B_i h \; z^{\alpha}] \\
&= e_{\emptyset} \otimes \Phi_C(D_i h - \tilde{C} B_i h) - \sum_{|\alpha|\geq 1} e_{\bar{\alpha}} \otimes \Phi_C(\tilde{C}(A_{\bar{\alpha}})^* B_i h) \\
&= e_{\emptyset} \otimes [(D_C)_i h - \gamma D_{*,A} B_i h] - \sum_{|\alpha|\geq 1} e_{\bar{\alpha}} \otimes \gamma D_{*,A} (A_{\bar{\alpha}})^* B_i h.
\end{aligned}$$

By equation (6.7) it follows that

$$\begin{aligned}
\tilde{M}_{\Phi_C} \Theta_{U,\tilde{U}}(z)(F_i h z^{\emptyset}) &= \Theta_{C,E}(e_{\emptyset} \otimes (D_E)_i h) \\
&= \Theta_{C,E} \tilde{\Phi}_E (F_i h z^{\emptyset}).
\end{aligned}$$

Case 2. $h \in \mathcal{H}^{\circ}$:

$$\begin{aligned}
\tilde{D} F_i h &= \sum_{j=1}^{d} D_j P_{\tilde{\mathcal{H}}} F_j^* F_i h = \sum_{j=1}^{d} D_j P_{\tilde{\mathcal{H}}} (\delta_{ij} I - E_j^* E_i) h \\
&= D_i P_{\tilde{\mathcal{H}}} h - \big(\sum_{j=1}^{d} D_j P_{\tilde{\mathcal{H}}} E_j^*\big) E_i h = -\tilde{C} A_i h
\end{aligned}$$

Second equality follows from equation (6.5) respectively. By equations (6.5) and (4.13) again we obtain

$$\begin{aligned}
\sum_{\beta\in\tilde{\Lambda},j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^* F_j^* F_i h \; z^{\beta j} &= \sum_{\beta\in\tilde{\Lambda},j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^* (\delta_{ij} I - E_j^* E_i) h \; z^{\beta j} \\
&= \sum_{\beta\in\tilde{\Lambda},j=1,\ldots,d} \tilde{C}(A_{\bar{\beta}})^* (\delta_{ij} I - A_j^* A_i) h \; z^{\beta j}.
\end{aligned}$$

So by equation (6.9) we have for all $i = 1, \ldots, d$ and $h \in \mathcal{H}^\circ$

$$\begin{aligned}
\tilde{M}_{\Phi_C}\Theta_{U,\tilde{U}}(z)(F_i h z^{\emptyset}) &= \tilde{M}_{\Phi_C}[-\tilde{C}A_i h\, z^{\emptyset} + \sum_{\beta\in\tilde{\Lambda}, j=1,\ldots,d} \tilde{C}(A_{\bar{\beta}})^*(\delta_{ij}I - A_j^*A_i)h\, z^{\beta j}] \\
&= -e_{\emptyset}\otimes\Phi_C(\tilde{C}A_i h) + \sum_{\beta\in\tilde{\Lambda}, j=1,\ldots,d} e_j\otimes e_{\bar{\beta}}\otimes\Phi_C(\tilde{C}(A_{\bar{\beta}})^*(\delta_{ij}I - A_j^*A_i)h) \\
&= -e_{\emptyset}\otimes\gamma D_{*,A}A_i h + \sum_{\beta\in\tilde{\Lambda}, j=1,\ldots,d} e_j\otimes e_{\bar{\beta}}\otimes\gamma D_{*,A}(A_{\bar{\beta}})^*(\delta_{ij}I - A_j^*A_i)h.
\end{aligned}$$

By equation (6.8) it follows that

$$\begin{aligned}
\tilde{M}_{\Phi_C}\Theta_{U,\tilde{U}}(z)(F_i h z^{\emptyset}) &= \Theta_{C,E}(e_{\emptyset}\otimes(D_E)_i h) \\
&= \Theta_{C,E}\tilde{\Phi}_E(F_i h z^{\emptyset}).
\end{aligned}$$

Hence we conclude that

$$\tilde{M}_{\Phi_C}\Theta_{U,\tilde{U}}(z) = \Theta_{C,E}\tilde{\Phi}_E.$$

□

For the generalised repeated interaction model Theorem 6.1 elucidates that the transfer function, which is a notion affiliated to the scattering theory, is identifiable with the characteristic function of the associated lifting. This establishes a strong connection between a model for quantum systems and the multivariate operator theory. Connections between them were also endorsed in other works like [Bh96], [Go04], [DG07] and [Go11], and this indicates that such approaches to quantum systems using multi-analytic operators is promising.

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